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F-CONVEX FUNCTIONS

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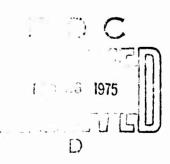Israel Institute of Technology, Haifa, Israel.

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F-CONVEX FUNCTIONS

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ABSTRACT

Let F be a family of functions: $R^n \to R$. A function:
$R^n \to R$ is called F-convex if it is supported, at each point,
by some member of F. For particular choices of F one
obtains the convex functions: $R^n \to R$ and the generalized
convex functions in the sense of Beckenbach. F-convex
functions are characterized and studied, retaining some
essential results of classical convexity.

F-CONVEX FUNCTIONS

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§1. INTRODUCTION

Let F be a family of functions: $R^n \to R$, depending on (n+1) parameters $\{x^*, \xi^*\} \in R^n \times R$. A function $f: R^n \to R$ is called F-convex if its graph is supported at each point by some member of F, see Definition 2.1. For particular choices of F , the F-convex functions reduce to the ordinary proper convex functions (Example 2.2) and the sub F-functions of Beckenbach (Example 2.3 and Proposition 2.4).

In this paper we study the basic properties of F-convex functions.

Sections 2 and 3 contain definitions and examples.

Section 4 gives first order conditions (so called because they involve only first derivatives and the "gradients" $\{x_f^*, \xi_f^*\}$ defined in 3.2) for F - convexity. For families $F \in \underline{A}$, see Definition 3.2, F-convexity is characterized in Theorem 4.2 by an analog of the gradient inequality. The remaining results in Section 4 are conditions for F-convexity or strict F-convexity, in terms of the mapping: $x \to \{x_f^*(x),\ \xi_f^*(x)\}$ or the F-gradient mapping: $x \to x_f^*(x)$, see Definition 3.3.

Second order conditions for F-convexity and strict F-convexity
are given in Section 5. These conditions involve the matrix

$$(5.1) \qquad H(x) = f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x); x)$$

which in the classical case reduces to the Hessian matrix (Example
5.2). The main results here are Theorems 5.1 and 5.5. An analog
of the differential inequality of Peixoto [14], characterizing
sub-F functions, is obtained as a special case (Example 5.3).

Section 6 deals with the monotonicity properties of the
F-gradient x_f^* of an F-convex function, where F belongs to
certain classes defined in 6.1. The derivative of x_f^* is computed
in Lemma 6.2, and the result is used, for the separable families
(6.10), to establish that x_f^* is a P_0-function [P-function] if f
is F-convex [strictly F-convex], see Theorem 6.4.

In a sequel paper we study the corresponding generalizations
of conjugacy and duality in the sense of Fenchel [16]. These results
involve a conjugate family F^*, and are hidden in the classical case
by the fact that there $F = F^*$.

§2. F-CONVEX FUNCTIONS: DEFINITIONS AND EXAMPLES

2.1 Definitions

Let F be a family of functions: $R^n \to R$ with common domain X

$$(2.1) \qquad X \triangleq \cap \{\text{dom } F: F \in F\}$$

and range

$$(2.2) \qquad \Xi \triangleq \cup \{\text{range } F: F \in F\} \quad .$$

Let f be a function: $R^n \to R$ with domain

$$(2.3) \qquad \text{dom } f \subset X$$

and let S be an open subset of dom f. Then f is called F-convex in S if for every $x \in S$, there exists an $F \in F$ such that

$$(2.4) \qquad f(x) = F(x) \quad \text{and} \quad f(z) \geq F(z) \quad \text{for all} \quad x \neq z \in S, \quad [1]$$

in which case F is called a support of f: S at x. The function f is called strictly F-convex in S if strict inequality holds in (2.4) for all $x \neq z \in s$.

If there is no need to specify S, for example if $S = \text{dom } f$, the above names are abbreviated by omitting S, e.g., F-convex, support of f at x, etc.

---

[1] The name F-convex function was used recently ([15], p.241) to denote the sub-F functions, see Example 2.2

2.2 Example

Let F be the family of affine functions: $R^n \to R$, i.e.,

$$(2.5) \qquad F = \{F(.) = \langle x^*, \cdot \rangle - \xi^*: x^* \in R^n, \xi^* \in R\} .$$

Then a function $f: R^n \to R$ is F-convex if and only if it is a proper convex function, i.e., a convex function whose epigraph is a non empty set containing no vertical lines, ([16], §4).

2.3 Example

Let F be a family of continuous functions: $R \to R$ with domain $X = (a,b)$ and such that

(B) For any two distinct points in X, say,

$$a < x_1 < x_2 < b$$

and any two real numbers $\{y_1, y_2\}$, there is a unique $F \in F$ satisfying

$$F(x_i) = y_i , \qquad (i = 1,2) .$$

We call such an F a Beckenbach family in (a,b). E.F. Beckenbach [1] called a function $f: (a,b) \to R$ a sub-F function if for any two points

$$a < x_1 < x_2 < b$$

the member of F, F_{12}, defined by

(2.6) $F_{12}(x_i) = f(x_i)$, $(i = 1,2)$,

satisfies

$$f(x) \leq F_{12}(x) , \qquad x_1 < x < x_2 .$$

M.M. Peixoto ([13],[14] Theorem 1) showed that if f is a sub-F
function and $a < x_0 < b$ then there exist two functions

$$F_i \in F , \qquad (i = 1,2) ,$$

such that

$$F_i(x_0) = f(x_0) , \qquad (i = 1,2) ,$$

$$F_2(x) \leq F_1(x) \leq f(x) , \qquad (a < x < x_0) ,$$

and

$$F_1(x) \leq F_2(x) \leq f(x) , \qquad x_0 < x < b .$$

(Furthermore, if the derivatives $f'(x_0)$, $F_1'(x_0)$ and $F_2'(x_0)$ exist,
they are equal). Thus both F_1 and F_2 support f at x_0 .

Therefore every sub-F function is F-convex. We will now prove
the converse for Beckenbach families F .

2.4 Proposition

Let F be a Beckenbach family in (a,b). Then a function
$f: (a,b) \to R$ is F-convex in (a,b) if and only if f is a sub-F
function.

Proof.

The proof of the "if" part was cited above.

To prove "only if" suppose f is not a sub-F function, i.e.,
there are three points

$$a < x_1 < x_0 < x_2 < b$$

such that the function $F_{12} \in F$, defined by (2.6), satisfies

$$(2.7) \qquad\qquad F_{12}(x_0) < f(x_0) \ .$$

Suppose that $F_0 \in F$ is a support of f at x_0 , i.e.,

$$(2.8) \qquad\qquad f(x_0) = F_0(x_0) \quad \text{and} \quad f(x) \geq F_0(x), \quad a < x < b \ .$$

From (2.6), (2.7) and (2.8) it follows that F_{12} and F_0 intersect
twice over the interval (a,b), contradicting (B). Therefore f is
not F-convex.   □

2.5 Example

Let $G(x,y,z)$ be a continuous function: $(a,b) \times R \times R \to R$, such that
(P1)   For each $\{x_0, y_0, y_0'\} \in (a,b) \times R \times R$, the differential equation

(2.9) $\qquad y'' = G(x,y,y')$, $\qquad (a < x < b)$,

has a unique solution $y = y(x)$ satisfying

(2.10) $\qquad y(x_0) = y_0$, $\qquad y'(x_0) = y_0'$.

(P2) $\qquad$ The solution of (2.9) is continuous with respect to the initial values y_0, y_0' .

(P3) $\qquad$ For any two points $\{x_i, y_i\} \in (a,b) \times R$ $(i = 1,2)$ with $x_1 \neq x_2$, there is a unique solution of (2.9) satisfying

(2.11) $\qquad y(x_i) = y_i$, $\qquad i = 1,2$.

Let F be the Beckenbach family of solutions of (2.9). M.M. Peixoto ([14] Theorem 2) showed that a function $f \in C^2(a,b)$ is a sub-F function if and only if

(2.12) $\qquad f'' \geq G(x,f,f')$, $\qquad a < x < b$.

2.6 Example

While sub-F functions are continuous ([1],[15] p. 242), an F-convex function need not be continuous in its domain, even if each $F \in F$ is continuous:

Let F be the family of functions: $R \to R$

$$F(x) = F(x^*, \xi^*; x) = \xi^* \sin(e^{x^*|x|} - 1)$$

depending on the two parameters

$$x^* \geq 0, \qquad 0 \leq \xi^* < 1.$$

Then the function

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is F-convex. Indeed, for every $x \neq 0$, the function $F = F(x^*, \xi^*; \cdot)$ defined by

$$x^* = \frac{\log(1 + \frac{\pi}{2})}{|x|}, \qquad \xi^* = 1$$

supports f at x. Also every $F \in F$ supports f at 0.

We show now that an F-convex function inherits from F its lower semi continuity.

2.7 Proposition

Let F be a family of l.s.c. (= lower semi continuous) functions and let f be F-convex in $\text{dom}\, f$. Then f is l.s.c. in $\text{dom}\, f$.

Proof.

Suppose f is not l.s.c.. Then there exists an $x \in \text{dom}\, f$ such that

(2.13) $$f(x) > \liminf_{y \to x} f(y).$$

Let $F \in F$ support f at x. Then

$$F(x) = f(x) > \liminf_{y \to x} f(y) \geq \liminf_{y \to x} F(y),$$
$$\text{by (2.4) and (2.13),}$$

contradicting the lower semi continuity of F.   □

2.8 Notes

For further generalizations of convexity see the surveys in ([2], Chapter 4), [4], and ([15], Chapter VIII).

For functions of several variables, the analogs of the sub F-functions are the subfunctions in particular the subharmonic functions; see [2] p. 146, [3] and [8], where applications to second order differential inequalities are surveyed.

§3. REQUIREMENTS ON F

3.1 General

With Examples 2.2, 2.3 and 2.5 as our motivation, we consider from now on only families F of functions $F: R^n \to R$ depending continuously on $n+1$ parameters

$$\{x^*, \xi^*\} \in X^* \times \Xi^*$$

where the sets of parameters X^* and Ξ^* are given subsets of R^n and R respectively. The general member of F is thus denoted by

$$(3.1) \qquad F(.) = F(x^*, \xi^*; \cdot), \quad (x^* \in X^*, \ \xi^* \in \Xi^*) ,$$

with function values

$$(3.2) \qquad F(x) = F(x^*, \xi^*; x), \qquad x \in X .$$

We assume that the mapping: $\{x^*, \xi^*\} \to F(x^*, \xi^*; \cdot)$ is one to one on $X^* \times \Xi^*$, i.e., $F(x^*, \xi^*; \cdot)$ is uniquely determined by $\{x^*, \xi^*\}$.

3.2 The class $\underset{\sim}{A}$

Let $D^k(X)$ denote the functions: $R^n \to R$ which are k times differentiable in X. If $F \subset D(X)$ we define the set

$$(3.3) \qquad Z \overset{\Delta}{=} \cup \left\{ \text{range} \begin{bmatrix} F \\ F_x \end{bmatrix} : F \in F \right\} \subset R \times R^n$$

where F_x is the gradient of F with respect to x.

A family F of differentiable functions is said to be in class $\underset{\sim}{A}$, denoted by $F \in \underset{\sim}{A}$, if for every $x \in X$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in Z$, the system

$$(3.4) \qquad\qquad \xi = F(x^*, \xi^*; x)$$

$$(3.5) \qquad\qquad y = F_x(x^*, \xi^*; x)$$

has a unique solution $\{x^*, \xi^*\} \in X^* \times \Xi^*$.

If $F \subset D(X)$ and if f and S are a function: $R^n \nrightarrow R$ and an open subset of dom f respectively, we denote by

$$(3.6) \qquad\qquad f \overset{S}{\approx} F$$

the facts

$$(D1) \qquad\qquad S \subset \text{dom } f \subset X$$

$$(D2) \qquad\qquad f \in D(S)$$

$$(D3) \qquad\qquad \text{range } \{ \begin{bmatrix} f(x) \\ f_x(x) \end{bmatrix} : x \in S \} \subset Z .$$

We abbreviate $f \overset{\text{dom } f}{\approx} F$ by $f \approx F$.

If $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \text{dom } f$ we denote by

$$(3.7) \qquad\qquad (x_f^*(x), \xi_f^*(x))$$

the unique solution of

$$(3.8) \qquad\qquad f(x) = F(x^*, \xi^*; x)$$

$$(3.9) \qquad\qquad f_x(x) = F_x(x^*, \xi^*; x) .$$

A family F of differentiable functions is said to be in **class** $\underset{\sim}{A}$, denoted by $F \in \underset{\sim}{A}$, if for every $x \in X$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in Z$, the system

$$(3.4) \qquad \qquad \xi = F(x^*, \xi^*; x)$$

$$(3.5) \qquad \qquad y = F_x(x^*, \xi^*; x)$$

has a unique solution $\{x^*, \xi^*\} \in X^* \times \Xi^*$.

If $F \subset D(X)$ and if f and S are a function. $R^n \cdot / R$ and an open subset of $\operatorname{dom} f$ respectively, we denote by

$$(3.6) \qquad \qquad f \overset{S}{\approx} F \quad . \quad .$$

the facts

$$(D1) \qquad \qquad S \subset \operatorname{dom} f \subset X$$

$$(D2) \qquad \qquad f \in D(S)$$

$$(D3) \qquad \qquad \operatorname{range} \left\{ \begin{bmatrix} f(x) \\ f_x(x) \end{bmatrix} : x \in S \right\} \subset Z .$$

We abbreviate $f \overset{\operatorname{dom} f}{\approx} F$ by $f \approx F$.

If $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \operatorname{dom} f$ we denote by

$$(3.7) \qquad \qquad (x_f^*(x), \xi_f^*(x))$$

the unique solution of

$$(3.8) \qquad \qquad f(x) = F(x^*, \xi^*; x)$$

$$(3.9) \qquad \qquad f_x(x) = F_x(x^*, \xi^*; x) .$$

3.3 The class $\underset{\sim}{C}$

A family F is said to be in class $\underset{\sim}{C}$, denoted by $F \in \underset{\sim}{C}$, if for every $\{x^*, x\} \in X^* \times X$ the function $F(x^*, \cdot; x)$ is a strictly decreasing function of $\xi^*, \xi^* \in \Xi^*$. In this case, we denote by $F^I(x, \cdot; x^*)$ the inverse function of $F(x^*, \cdot; x)$. It satisfies the identity

$$(3.10) \qquad \xi = F(x^*, F^I(x, \xi; x^*); x), \xi \in \Xi \quad .$$

If $F \in \underset{\sim}{A} \cap \underset{\sim}{C}$, $f \approx F$ and $x \in \text{dom} f$, then (3.8) gives

$$(3.11) \qquad \xi^* = F^I(x, f(x); x^*)$$

which, substituted in (3.9), gives

$$(3.12) \qquad f_x(x) = F_x(x^*, F^I(x, f(x); x^*); x) \quad .$$

The unique solution of (3.12) is then called the F-gradient of f at x, and is denoted by $x_f^*(x)$.

3.4 Example

Let F be the family (2.5) of affine functions: $R^n \to R$. Then

(a) $F \subset D(R^n)$, $F_x(x^*, \xi^*; x) = x^*$ for every $F \in F$ and $x \in R^n$, and (3.3) gives $Z = R \times R^n$.

(b) $F \in \underset{\sim}{A}$. For every $x \in R^n$ and $\begin{bmatrix} \xi \\ y \end{bmatrix} \in R \times R^n$, the unique solution of (3.4)-(3.5) is

$$x^* = y , \quad \xi^* = \langle y, x \rangle - \xi .$$

(c) $F \in \underset{\sim}{C}$.

(d) $f \approx F$ means that $f \in D(\text{dom } f)$.

(e) If $f \approx F$ then for every $x \in \text{dom } f$

(3.13) $$x_f^*(x) = f_x(x), \quad \xi_f^*(x) = \langle f_x(x), x \rangle - f(x) .$$

Thus the F-gradient of f, x_f^* , coincides here with its ordinary gradient f_x .

3.5 Example

Let ϕ be a given function: $X^* \times X \to R$ and let the family F consist of the functions $F(x^*, \xi^*; \cdot)$, $\{x^*, \xi^*\} \in X^* \times \Xi^*$, with values

(3.14) $$F(x^*, \xi^*; x) = \phi(x^*, x) - \xi^* , \quad x \in X.$$

Then:

(a) $F \in \underset{\sim}{A}$ if and only if the following two conditions hold:

(a1) $\phi(x^*, \cdot) \in D(X)$ for every $x^* \in X^*$.

(a2) For every $x \in X$, $y \in \underset{x^*}{\cup} \text{range } \phi_x(x^*, \cdot)$, the system

$$y = \phi_x(x^*, x)$$

has a unique solution x^* .

(b) $F \in \underset{\sim}{C}$.

(c)   Let $F \in \underset{\sim}{A}$, $f \approx F$ and $x \in \mathrm{dom}\, f$. Then the F-gradient of f

at x , $x_f^*(x)$, is the unique solution x^* of

(3.15) $f_x(x) = \phi_x(x^*, x)$.

Also

(3.16) $\xi_f^*(x) = \phi(x_f^*(x), x) - f(x)$.

A concrete example is the following family F defined by

$$F(x^*, \xi^*; x) \triangleq \sum_{i=1}^{n} x_i^* \phi^i(x_i) + \frac{1}{\sum\limits_{i=1}^{n} x_i^* \phi^i(x_i)} - \xi^*$$

$$X^* = R_+^n , \quad \Xi^* = R, \quad X = \cap \, \mathrm{dom}\, \phi^i$$

where for every $i = 1, 2, \ldots, n$, $\phi^i : R \to R_+$ is differentiable and

$\phi_{x_i}^i > 0$.   The condition $f \approx F$ for this case is

$$f_{x_i} > 0 \qquad i = 1, 2, \ldots, n$$

or

$$f_{x_i} < 0 \qquad i = 1, 2, \ldots, n .$$

The F-gradient is

$$x_f^*(x) = \left(\frac{t^2(x)}{t^2(x) - 1}\right) \begin{bmatrix} f_{x_1}(x)/\phi_{x_1}^1(x_1) \\ \vdots \\ f_{x_n}(x)/\phi_{x_n}^n(x_n) \end{bmatrix} ,$$

where

$$t(x) \triangleq \frac{1}{2} \left\{ \sum_{i=1}^{n} \frac{f_{x_i}}{\phi_{x_i}^i} \phi^i + \left[\left(\sum_{i=1}^{n} \frac{f_{x_i}}{\phi_{x_i}^i} \phi^i \right) + 4 \right]^{1/2} \right\} .$$

§4. FIRST ORDER CONDITIONS FOR F-CONVEXITY

In this section we give first order conditions (so-called because they involve only first derivatives and the "gradients" $\{x_f^*, \xi_f^*\}$ of f, see (3.7)) for F-convexity, for families F in class $\underset{\sim}{A}$. These conditions use the extremal property of the supports implied by the inequality (2.4). First we require

4.1 Lemma

Let $F \in \underset{\sim}{A}$, $f: R^n \to R$, and let $f \overset{S}{\approx} F$. If $f: S$ is supported (by some $F \in F$) at a point $x \in S$, then

$$(4.1) \qquad F(x_f^*(x), \xi_f^*(x); \cdot)$$

is the unique support of f at x.

Proof.

Let $F(x_0^*, \xi_0^*; \cdot) \in F$ support $f: S$ at x, i.e.,

$$(4.2) \qquad h(z) \overset{\Delta}{=} f(z) - F(x_0^*, \xi_0^*; z) \geq 0, \quad \forall z \in S,$$

and

$$(4.3) \qquad h(x) = f(x) - F(x_0^*, \xi_0^*; x) = 0.$$

Therefore $h(z)$ is minimized, in S, by $z = x$. Since S is open, this implies that x is a critical point of h, i.e.,

$$(4.4) \qquad h_z(x) = f_x(x) - F_x(x_0^*, \xi_0^*; x) = 0.$$

Since $F \in \underset{\sim}{A}$, a comparison of (4.3)-(4.4) and (3.8)-(3.9) shows that

$$\{x_0^*, \xi_0^*\} = \{x_f^*(x), \xi_f^*(x)\}$$

proving that (4.1) is the unique support at x. □

4.2 Theorem

Let $F \in \underset{\sim}{A}$, $f: R^n \to R$, and $f \overset{S}{\approx} F$. Then f is F-convex in S if and only if for every $x \in S$

(4.5) $\qquad f(z) \geq F(x_f^*(x), \xi_f^*(x); z), \quad \forall\, x \neq z \in S$.

Furthermore, f is strictly F-convex in S if and only if for every $x \in S$

(4.6) $\qquad f(z) > F(x_f^*(x), \xi_f^*(x); z), \quad \forall\, x \neq z \in S$.

Proof.

If. From (4.5) and (3.8) it follows, for any $x \in S$, that the function (4.1) supports $f: S$ at x. It is the unique support if (4.6) holds.

Only if. Let f be F-convex in S. Then, by Lemma 4.1, for any $x \in S$, the function (4.1) is the unique support of $f: S$ at x. The inequality (4.5) then follows from (2.4). Similarly (4.6) follows from the strict F-convexity of f. □

4.3 Example

Let F be the family (2.5) of affine functions: $R^n \to R$,

$$F = \{F(x^*, \xi^*; \cdot) = \langle x^*, \cdot \rangle - \xi^* : x^* \in R^n, \xi^* \in R\} .$$

Then, using (3.13), the inequality (4.5) reduces to

$$f(z) \geq \langle f_x(x), z - x \rangle + f(x), \qquad \forall \; x \neq z \in S,$$

the classical gradient inequality.

4.4 Corollary

(a) Let $F \in \underset{\sim}{A}$, and let $f: R^n \to R$, $f \overset{S}{\gtrless} F$, be F-convex in S. Then f is strictly F-convex in S if and only if the mapping

$$(4.7) \qquad x \to \{x_f^*(x), \xi_f^*(x)\}$$

is one to one on S.

(b) Let, in addition, $F \in \underset{\sim}{C}$. Then f is strictly F-convex in S if and only if the mapping

$$(4.8) \qquad x_f^*: \; x \to x_f^*(x)$$

is one to one on S.

Proof.

From Lemma 4.1 it follows, for every $x \in S$, that the function (4.1) is the unique support of $f: S$ at x. By definition, f is

strictly F-convex in S if, and only if, every support of f: S supports f at exactly one point of S . This is equivalent to the mapping (4.7) being one to one on S .

To prove the last part, note that the additional hypothesis $F \in \underset{\sim}{C}$ implies

$$(4.9) \qquad [x \overset{1:1}{\to} x_f^*(x) \text{ on } S] \longleftrightarrow [x \overset{1:1}{\to} \{x_f^*(x), \xi_f^*(x)\} \text{ on } S].$$

Indeed, the implication $\longrightarrow$ is always true. Conversely, suppose that x_f^* is not one to one on S , i.e., there exist $x_1, x_2 \in S$, $x_1 \neq x_2$, such that

$$(4.10) \qquad x_f^*(x_1) = x_f^*(x_2) \overset{\Delta}{=} x_0^* .$$

Let $\xi_i^* \overset{\Delta}{=} \xi_f^*(x_i) = F^I(x_i, f(x_i); x_0^*)$ and let

$$F^i(\cdot) \overset{\Delta}{=} F(x_0^*, \xi_i^*; \cdot), \qquad (i = 1,2) .$$

Then

$$F^i(x_i) = f(x_i), \qquad F_{::}^{\cdot}(x_i) = f_x(x_i) , \qquad i = 1,2 .$$

Hence by Theorem 4.2, F^i supports f at x_i . If $\xi_1^* = \xi_2^*$, then this and (4.10) contradicts the fact that (x_f^*, ξ_f^*) is 1:1, established earlier. Thus suppose that $\xi_1^* > \xi_2^*$. This implies, since $F \in \underset{\sim}{C}$, that $F^1(z) < F^2(z) \; \forall \, z \in S$. In particular

$$F^2(x_1) > F^1(x_1) = f(x_1)$$

contradicting the fact that F^2 is a support. $\square$

4.5 Theorem

Let $F \in \underset{\sim}{A} \cap \underset{\sim}{C}$, $f: R^n \to R$, and $f \overset{S}{\approx} F$. Then f is strictly F-convex in S if the following two conditions hold.

(a) The mapping x_f^* is one to one on S.

(b) For every $x \in S$ and for every sequence $\{z_k\} \subset S$ which either converges to a point $y \in \text{bdry } S$ or $|z_k| \to \infty$ there exists an $\hat{x} \in S$ such that

$$(4.11) \qquad \limsup_{k \to \infty} \{F^I(z_k, f(z_k), x_f^*(x)) - F^I(\hat{x}, f(\hat{x}), x_f^*(x))\} \leq 0$$

where F^I is defined in §3.3.

Proof.

For any $x \in S$ consider the function

$$(4.12) \qquad T(z) \triangleq F^I(z, f(z); x_f^*(x)).$$

We show first that $z = x$ is a critical point of T. Differentiating the identity

$$(4.13) \qquad F(x^*, F^I(y, f(y); x^*); y) - f(y) = 0$$

with respect to y we get

$$(4.14) \quad F_x(\cdot, \cdot; \cdot) + F_{\xi^*}(\cdot, \cdot; \cdot)[F_x^I(y, f(y); x^*) + F_\xi^I(y, f(y); x^*)f_x(y)] - f_x(y) = 0$$

where

$$(\cdot, \cdot; \cdot) = (x^*, F^I(y, f(y); x^*); y).$$

Now $F_{\xi*} \neq 0$, since $F \in \underset{\sim}{C}$. Therefore, for $y = x$ and $x^* = x_f^*(x)$, it follows from (4.14) and (3.12) that

$$(4.15) \qquad F_x^I(x, f(x); x_f^*(x)) + F_\xi^I(x, f(x); x_f^*(x)) \, f_x(x) = 0$$

which, by (4.12), is the same as $T_z(x) = 0$, proving that $z = x$ is critical.

Moreover, $z = x$ is the unique critical point of T in S. For suppose that $x \neq x' \in S$ is another critical point of T, i.e.

$$T_z(x') = F_x^I(x', f(x'); x_f^*(x)) + F_\xi^I(x', f(x'); x_f^*(x)) f_x(x') = 0$$

implying that for $y = x'$ and $x^* = x_f^*(x)$, (4.14) reduces to

$$F_x(x_f^*(x), F^I(x', f(x'); x_f^*(x)); x') - f_x(x') = 0$$

which, together with (3.12), implies that

$$x_f^*(x') = x_f^*(x) ,$$

contradicting (a).

We show next that

$$(4.16) \qquad \sup\{T(z): z \in S\} = T(x) .$$

Indeed if this supremum occurs at some $z = y \in \text{bdry } S$ or if a supremizing sequence $\{z_k\}$ is such that $|z_k| \to \infty$ then the supremum is also attained at $\hat{x} \in S$, by (4.11). Therefore $z = \hat{x}$ is a critical point of T, proving that $\hat{x} = x$, since the latter

is the unique critical point in S , and therefore (4.16) becomes

$$F^I(x,f(x); x_f^*(x)) > F^I(z,f(z); x_f^*(x)) , \quad \forall \ x \neq z \in S ,$$

which is the same as

$$f(z) > F(x_f^*(x), F^I(x,f(x); x_f^*(x)); z), \quad \forall \ x \neq z \in S ,$$

proving that f is strictly F-convex in S , by theorem 4.2.   □

4.6 Example

Consider the family

$$F = \{\phi(x^*,\cdot) - \xi^*: \ x^* \in X^*, \ \xi^* \in \Xi^* \} ,$$

of Example 3.5 and let $F \in \underline{A}$, $f: R^n \to R$, and $f \overset{S}{\gtrsim} F$. Then condition (b) of Theorem 4.5 follows from

(b1) For every $x^* \in \text{range}\{x_f^*(x): x \in S\}$ and every sequence $\{z_k\}$ as in Theorem 4.5(b),

$$(4.17) \qquad\qquad \liminf_{k \to \infty}\{f(z_k) - \phi(x^*,z_k)\} = +\infty . \qquad\qquad □$$

In particular, if

$$S = \text{dom} \ f = X = R^n$$

and

$$(4.18) \qquad\qquad \limsup_{|x| \to \infty} \frac{\phi(x^*,x)}{|x|} < \infty , \quad \forall \ x^* \in \text{range} \ x_f^* .$$

then condition (b) of Theorem 4.5 is satisfied if

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty \ .$$

Note that (4.18) is trivially satisfied by the family F of affine functions. Hence, a differentiable function $f: R^n \to R$ is strictly convex if the following two conditions hold.

(a) The mapping

$$x \to f_x(x)$$

is one to one on R^n.

(b) $\qquad\qquad\qquad\qquad \lim_{|x| \to \infty} \frac{f(x)}{|x|} = \infty \ .$ $\qquad\qquad$ ☐

As a concrete example of condition (b1) let F be the family of functions: $R^2 \to R$ given by

(4.19) $\qquad F(x^*, \xi^*; \ x) = x_1^* \ e^{-x_1} + x_2^* x_2 \ e^{-x_1} - \xi^*$

with $X = X^* = R^2, \quad \Xi^* = R$.

Consider the function $f: R^2 \to R$

(4.20) $\qquad f(x) = \frac{1}{2} e^{2x_1} + \frac{1}{2} x_2^2 e^{-x_1}$

with $\mathrm{dom} \ f = R^2$. Then f is F-convex in R^2 since:

(a) The F-gradient

$$x_f^*(x) = \begin{bmatrix} -e^{3x_1} - \frac{1}{2} x_2^2 \\ \\ x_2 \end{bmatrix}$$

is one-to-one, and

(4.21) $$\text{range } x_f^* = \{(x_1^*, x_2^*) \in R^2 : x_1^* + \tfrac{1}{2} x_2^{*2} < 0\}.$$

(b) $$f(z) - \phi(x^*, z) = (\tfrac{1}{2} e^{2z_1} + \tfrac{1}{2} z_2^2 e^{-z_1}) - (x_1^* e^{-z_1} + x_2^* z_2 e^{-z_1}) =$$

$$= \tfrac{1}{2} e^{2z_1} - (x_1^* + \tfrac{1}{2} x_2^{*2}) e^{-z_1} + \tfrac{1}{2} (z_2 - x_2^*)^2 e^{-z_1}$$

by (4.21) the coefficients of all exponents are positive and hence

$$\lim_{|z| \to \infty} [f(z) - \phi(x^*, z)] = \infty \qquad \forall\, x^* \in \text{range } x_f^* .$$

§5. SECOND ORDER CONDITIONS FOR F-CONVEXITY

In this section we collect second order conditions (involving second derivatives) for F-convexity.

5.1 Theorem

Let $F \in \underline{A} \cap D^2(X)$, $f: R^n \to R$, $f \overset{S}{\approx} F$ and $f \in D^2(S)$. Then:

(a) f is F-convex in S only if, for every $x \in S$, the matrix

$$(5.1) \qquad H(x) \overset{\Delta}{=} f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x); x)$$

is positive semi definite.[1]

(b) Let S be convex and let f and each $F \in F$ be twice continuously differentiable in S. Then f is F-convex in S if

$$(5.2) \qquad \langle y, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_f^*(x); x+sy))y\,ds \rangle \geq 0 \ ,$$

for every $x \in S$ and $y \in S - x$.

If strict inequality holds in (5.2), F is strictly F-convex in S.

Proof.

(a) Let f be F-convex in S. Then, for any $x \in S$, the function

$$(5.3) \qquad h(z) \overset{\Delta}{=} f(z) - F(x_f^*(x), \xi_f^*(x); z)$$

---

[1] A matrix $H \in R^{n \times n}$ is called here <u>positive semi definite</u> if

$$\langle Hz, z \rangle \geq 0, \quad \forall z \in R^n .$$

We do not mean by this that H is symmetric.

satisfies

(5.4) $\qquad h(x) = 0, \qquad h_z(x) = 0, \qquad$ by $(3.8)-(3.9)$,

and

$$h(z) \geq 0, \qquad \forall\, z \in S, \quad \text{by Theorem 4.2.}$$

Therefore $z = x$ minimizes h in S. Since S is an open set ,
it follows that

$$h_{zz}(x) = H(x)$$

is positive semi-definite.

(b) The function h of (5.3) satisfies

$$h(z) = h(z) - h(x) - \langle h_z(x),\, z - x \rangle, \quad \text{by (5.4)},$$

$$= \langle (h_z(x + t(z-x)) - h_z(x)),\, z-x \rangle, \quad \text{for some } 0 < t < 1,$$
$$\text{by a mean value theorem ([12], Theorem 3.2.2)},$$

$$= \langle z - x,\, (\int_0^1 h_{zz}(x + st(z-x))ds)\, t(z-x) \rangle,$$
$$\text{by a mean value theorem ([12], Theorem 3.2.7)},$$

$$= \frac{1}{t} \langle y,\, \int_0^1 (f_{xx}(x+sy) - F_{xx}(x_f^*(x), \xi_f^*(x); x+sy))y\, ds \rangle,$$
$$\text{where } y = t(z-x).$$

Thus, (5.2) implies that

(5.5) $\qquad\qquad h(z) \geq 0, \qquad \forall\, z \in S,$

proving that f is F-convex in S, by Theorem 4.2.

Similarly, strict inequality in (5.2) implies strict inequality in (5.5), hence strict F-convexity. □

5.2 Example

Let F be the family (2.5) of affine functions: $R^n \to R$. Then the matrix $H(x)$ of (5.1) reduces to the Hessian of f

$$H(x) = f_{xx}(x)$$

and Theorem 5.1 gives the classical conditions for convexity in terms of the Hessian.

5.3 Example

Let F be the Beckenbach family of solutions of the second order differential equation

$$(2.9) \qquad y'' = G(x,y,y'), \qquad (a < x < b) ,$$

discussed in Example 2.5. Then (5.1) becomes

$$H = f'' - G(x,f,f') .$$

Now, suppose that $F \subset C^2(X)$, $f \in C^2(S)$, then $H(x) > 0$ implies $H(x + sy) > 0$ for $0 < s < 1$ and y sufficiently close to x. Thus (5.2) is a strict inequality in some neighborhood of x, and we conclude that f is, locally, strictly F-convex. By proposition 2.4 this implies that f is locally strictly sub-F, which by ([1]

Theorem 7) implies that f is sub-F globally in (a,b). This
result is the analog of [14], Theorem 3. To get the analogous
result of ([14] Theorem 1), we need the implication $H(x) \geq 0 \longrightarrow$
$H(x + sy) \geq 0$, for $0 < s < 1$ and y sufficiently close to x, for
which Peixoto's additional requirement, (P2) of Example 2.5, is
needed (see Peixoto's proof of [14] Lemma 1).

5.4 Definition

A mapping $T: R^n \to R^n$ is called one to one on R^n if

(a) $x, y \in R^n$, $x \neq y \Rightarrow T(x) \neq T(y)$.

(b)   The inverse images $T^{-1}(B)$ of bounded sets $B \subset R^n$ are
bounded.

5.5 Theorem

Let $F \in \underset{\sim}{A} \cap \underset{\sim}{C} \cap C^2(R^n)$, $f: R^n \to R$, $f \in C^2(R^n)$ and $f \approx F$.
Then f is strictly F-convex in R^n if the following two conditions
hold

(a)   The mapping x_f^* is one to one on R^n.

(b)   For every $x \in R^n$, the matrix

$$(5.6) \qquad H(x) = f_{xx}(x) - F_{xx}(x_f^*(x),\ F^I(x, f(x);\ x_f^*(x));\ x)$$

is positive definite. Conversely, if f is strictly F-convex in R^n
then (a) holds and the matrix $H(x)$ is positive semi-definite for
every $x \in R^n$.

Proof.

First we note, by (3.11), that (5.6) and (5.1) are the same.

For any $x \in R^n$ consider now the function

(4.12)
$$T(z) = F^I(z, f(z); x_f^*(x)) .$$

As in the proof of Theorem 4.5 it follows from (a) that $z = x$ is the unique critical point of T in R^n.

Differentiating the identity (4.13) twice with respect to y we get, by using (4.15) and (3.12),

(5.7)
$$T_{zz}(x) = \frac{1}{F_{\xi^*}} H(x)$$

(where F_{ξ^*} is evaluated at $\{x_f^*(x), F^I(x, f(x); x_f^*(x)); x\}$). From (5.7), (b) and $F \in \underset{\sim}{C}$ it follows that $T_{zz}(x)$ is negative definite. Therefore $z = x$ is an isolated local maximizer of T, and its unique critical point in R^n.

Thus, by Leighton's Theorem [9], see also [17], $z = x$ is the global maximizer of T, i.e.,

$$F^I(x, f(x); x_f^*(x)) > F^I(z, f(z); x_f^*(x)), \quad \forall x \neq z \in R^n ,$$

which is the same as

$$f(z) > F(x_f^*(x), F^I(x, f(x); x_f^*(x)); z), \quad \forall x \neq z \in R^n ,$$

proving that f is strictly F-convex in R^n by Theorem 4.2. $\square$

If f is strictly F-convex in R^n then (a) and (b) follow from Corollary 4.4 and Theorem 5.1(a) respectively.

5.6 Example

Let F and f be given by (4.19) and (4.20) respectively. Then the matrix (5.1) is positive definite

$$H(x) = \begin{bmatrix} 3e^{2x_1} & 0 \\ 0 & e^{-x_1} \end{bmatrix}$$

and f is strictly F-convex in R^2, by Theorem 5.5.

§6. MONOTONICITY OF F-GRADIENTS

In this section we prove monotonicity results for the F-gradien
x_f^* of an F-convex function. We recall that a mapping $g: R^n \to R^n$
is a P-function [P_0-function] if for every $x,y \in \text{dom } g$, $x \neq y$, there
is an index $k = k(x,y) \in \{1,2,\ldots,n\}$ such that

$(x_k - y_k)(g_k(x) - g_k(y)) > 0$ $[(x_k - y_k)(g_k(x) - g_k(y)) \geq 0$ and
$x_k \neq y_k]$, see [10]. In particular, a mapping $g: R^n \to R^n$ is monoton
[strictly monotone] if for every $x,y \in \text{dom } g$, $x \neq y$, we have
$\langle x-y, g(x) - g(y)\rangle \geq 0$ $[\langle x-y, g(x) - g(y)\rangle > 0]$. We also require
the following

6.1 Definitions

A family F is said to be in class A_1, denoted by $F \in A_1$,
if $F \in A$ and for every $(x^*, \xi^*; x) \in X^* \times \Xi^* \times X$ the derivatives in
(6.1) are continuous and the matrix

$$(6.1) \qquad J(x^*,\xi^*; x) = \begin{bmatrix} F_{\xi^*}(x^*,\xi^*; x) & F_{x^*}^T(x^*,\xi^*; x) \\ F_{\xi^* x}(x^*,\xi^*; x) & F_{x^* x}(x^*,\xi^*; x) \end{bmatrix} \qquad 1)$$

is nonsingular, say

---

[1] This matrix is the Jacobian matrix of the function
$$\begin{bmatrix} F(\cdot,\cdot; x) \\ F_x(\cdot,\cdot; x) \end{bmatrix} ,$$
see (3.4)-(3.5).

(6.2) $$\det J(x^*, \xi^*; x) < 0 .$$

A family F is said to be in <u>class</u> A_2, denoted by $F \in A_2$, if $F \in A_1$ and for every $x \in X$ the matrix

(6.3) $$J_0(x) \triangleq \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^*x} - F_{\xi^*x} F_{x^*}^T] ,$$

where all derivatives are evaluated at $\{x_f^*(x), \xi_f^*(x); x\}$, is positive definite.

6.2 Lemma

Let $F \in A_1 \cap C$, $f: R^n \to R$, $f \overset{S}{\approx} F$ and let f and each $F \in F$ be twice continuously differentiable in S. Then, for every $x \in S$,

(6.4) $$D_x x_f^*(x) = J_0(x)^{-1} H(x)$$

where $D_x x_f^*(x)$ denotes the derivative of x_f^* at x and J_0 and H are given by (6.3) and (5.1) respectively.

Proof.

For any $x \in S$ consider the system

(3.8) $$F(x^*, \xi^*; x) - f(x) = 0$$

(3.9) $$F_x(x^*, \xi^*; x) - f_x(x) = 0$$

which, since $F \in A$, has a unique solution $\{x_f^*(x), \xi_f^*(x)\}$. The implicit function theorem, applicable since $F \in A_1$, then gives

$$(6.5) \quad \begin{bmatrix} D_x \xi_f^*(x) \\ D_x x_f^*(x) \end{bmatrix} = \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}^{-1} \begin{bmatrix} f_x(x) - F_x(x_f^*(x), \xi_f^*(x); x) \\ f_{xx}(x) - F_{xx}(x_f^*(x), \xi_f^*(x); x) \end{bmatrix}$$

where the derivatives

$$\begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix}$$

are evaluated at $\{x_f^*(x), \xi_f^*(x); x\}$.

Using (3.9) and (5.1), we rewrite (6.5) as

$$(6.6) \quad F_{\xi^*} D_x \xi_f^*(x) + F_{x^*}^T D_x x_f^*(x) = 0$$

$$(6.7) \quad F_{\xi^* x} D_x \xi_f^*(x) + F_{x^* x} D_x x_f^*(x) = H(x) .$$

Now $F_{\xi^*} \neq 0$ since $F \in \underset{\sim}{C}$. Eliminating $D_x \xi_f^*(x)$ from (6.6) and substituting in (6.7) gives

$$H(x) = \frac{1}{F_{\xi^*}} [F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] D_x x_f^*(x) .$$

The proof is completed by showing that the matrix

$$[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T]$$

is nonsingular, which follows since

$$(6.8) \quad \det[F_{\xi^*} F_{x^* x} - F_{\xi^* x} F_{x^*}^T] = F_{\xi^*}^{n-1} \det \begin{bmatrix} F_{\xi^*} & F_{x^*}^T \\ F_{\xi^* x} & F_{x^* x} \end{bmatrix} ,$$

by Sylvester's identity ([7], Section II.3),

$$\neq 0, \text{ since } F \in \underset{\sim}{C} \cap \underset{\sim}{A_1} . \qquad \Box$$

6.3 Example

Let F be the family (2.5) of affine functions: $R^n \to R$. Then

$$x_f^*(x) = f_x(x), \qquad \text{by Example 3.4,}$$

$$J_0(x) = I \qquad \text{by (6.3) since } F_{x^*x} = I, \quad F_{\xi^*x} = 0$$

and (6.4) reduces to the obvious

$$(6.9) \qquad D_x f_x(x) = f_{xx}(x) \; .$$

If f is a convex [strictly convex] differentiable function, then its gradient f_x is monotone [strictly monotone] in $\text{dom} f$. This is an immediate consequence of the gradient inequality (Example 4.3), and Theorem 4.2. Alternatively and less directly, the monotonicity of f_x can be shown to follow from (6.9) and the fact that f_{xx} is positive semi definite, see, e.g. [12], Theorem 5.4.3. Two other cases in which the factorization (6.4) is used to establish a monotonicity property of the F-gradient x_f^*, will now be given.

6.4 Theorem

Let $F \in A_2 \cap C^2(X)$ where $X = I_1 \times I_2 \times \ldots \times I_n$ is the product of open intervals $I_i \subset R$, $(i = 1, \ldots, n)$. Let each $F \in F$ be of the form

$$(6.10) \qquad F(x^*, \xi^*; x) = \sum_{i=1}^{n} F^i(x_i^*, x_i) - \xi^*$$

where $F^1(x_i^*, \cdot): I_i \to R$ $(i = 1, 2, \ldots, n)$. Let $f: R^n \to R$ be F-convex [strictly F-convex] with dom $f \supset X$ and $f \in C^2(X)$. Then x_f^* is a P_0-function [P-function] in X.

Proof.

From (6.10), (6.3) and $F \in A_2$ it follows that

$$J_0(x) = F_{x^*x}$$

a diagonal, positive definite matrix. From (6.4) and Theorem 5.1(a) it therefore follows, for an F-convex function f, that $D_x x_f^*(x)$ is a P_0-matrix, (see [5],[6]), proving that x_f^* is a P_0-function, by [10], Corollary 5.3.

If f is strictly F-convex, then, by Corollary 4.4(b) (applicable since $F \in C$), it follows for any $x, y \in X$, $x \neq y$, that there is a $k = k(x,y) \in \{1, 2, \ldots, n\}$ such that

$$x_k \neq y_k \quad \text{and} \quad x_f^*(x)_k \neq x_f^*(y)_k ,$$

proving that x_f^* is a P-function. $\square$

A special case of Theorem 6.4 is the following, one dimensional result:

6.5 Corollary

Let $F \in A_1 \cap C$ be a family of functions: $R \to R$, let $f: R \to R$, S an open subset of dom f, and let f and each $F \in F$ be twice continuously differentiable in S. If f is F-convex in S then x_f^* is a nondecreasing function in S.

Proof.

Using (6.3), (6.8) and (6.1) we write

(6.11)
$$J_0(x) = \frac{1}{F_{\xi^*}} \det J(x_f^*(x), \xi_f^*(x); x)$$

$$> 0 , \text{ by (6.2) and } F \in \underset{\sim}{C}.$$

Therefore

$$\frac{d}{dx} x_f^*(x) \geq 0 , \text{ by (6.4) and Theorem 5.1(a).} \qquad \square$$

6.6 Corollary

Let F, f and S be as in Corollary 6.5, where S is an interval (a,b). If

$$f''(x) > F_{xx}(x_f^*(x), \xi_f^*(x), x), \quad x \in S ,$$

then f is strictly F-convex.

Proof.

From (6.4) and (6.11) we infer that x_f^* is 1:1 on (a,b). As in the proof of Theorem 5.6 this implies that $z = x$ is a local minimizer of $h(z) \triangleq f(z) - F(x_f^*(x), \xi_f^*(x); z)$ and that no other critical point exists in (a,b). Hence $z = x$ is the unique global minimizer of h(z), which was previously shown to be equivalent to the strict F-convexity of f . $\qquad \square$

6.7 Corollary

Let F be as in Theorem 6.4, with $X = R^n$. A function $f: R^n \to R$ with $\text{dom } f = R^n$, $f \in C^2(R^n)$, $f \approx F$ is strictly F-convex, if the matrix $H(x)$ is positive definite.

Proof.

Follows from (6.4) and Theorem 5.5. $\qquad\qquad\Box$

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